The object of Dr. Wiegand is by no means confined to the law of Invalidity amongst Railway Officials. He is well aware of the importance of Invalidity Assurance in general, and he has no doubt that Life Assurance Offices will willingly cultivate this branch, as soon as they possess the necessary data for its computation. An Invalidity Table of Railway Officials will, in Dr. Wiegand's opinion, enable them to grant Invalidity Assurance to persons in any occupation, as the Invalidity to be expected in other pursuits is less than with Railway Officials, the Railway Service requiring so vigorous a constitution, that it must pension off its Officials, as invalids, in a state of health which would not entitle them to be considered as invalid in another occupation.

It is scarcely necessary to add that Dr. Wiegand, while he puts at the head of his researches the Invalidity, has not neglected the Mortality of Railway Officials. He says (page 4), "It is nearly an axiom with Life "Assurance Societies that railway Officials are subject to a very high rate of "mortality. Constant reports of railway accidents, where so and so many "officials have perished, must create the persuasion that this class has a very "low expectation of life. That is the reason why some Offices entirely "refuse to assure Railway Officials, and others only accept them with a very "considerable extra premium. But there exists no real measure for the "extra risk founded upon experience, and each Office is guided only by the "vague feelings of its manager. This state of things is unreasonable. If "there exists an extra risk, its true value must be found out by observation."

With reference to Mortality, Dr. Wiegand has examined the materials hitherto furnished to him by the Railway Societies; and although the data are by no means numerous enough to establish a general truth, the results of his examination are important and interesting.

Among 10952 Railway Officials under observation during a year, of whom 2193 belonged to the train officials, the mortality has been 124 and 25 respectively. According to the Experience table it ought to have been 134·2179 and 24·528.

Among 73379 Railway Officials under observation 415 cases of Invalidity have occurred, while, according to Dr. Wiegand's Hypothesis, there would have been 401·789 invalids.

The coincidence is surprising, but only an examination based on more numerous facts will decide about the real law.

Hamburg, May, 1869. WILHELM LAZARUS.

Notes on Newton's Formulæ for Interpolation.

I.

NEWTON, in a celebrated lemma (*Princ. Phil. Nat. Math.*, Lib. iii., *Lemma* v.), proposed and solved the fundamental problem of interpolation by finite differences; and thereby, to say the least, gave us the foundation of the theory of differences.

In the *Methodus Differentialis*, printed in 1715, he treats the same subject more at length; but after a careful examination, I cannot but think that this little treatise was written many years before the said lemma, which bears all appearance of being the ripe fruit of his researches on this matter.

On the other hand, the *Methodus Differentialis* has a great interest of its own, as showing how he dealt with such problems, beginning in the most direct and elementary way, and then—when he had overcome the first difficulties—going on so fast and with such large steps, that it requires no slight attention to follow him.

His solution of the problem is certainly as general, as elegant, and as simple, as we might expect from *summus** Newton; but alas,—he has given no proof of it! At least his commentators tell us so, and among them were men of great knowledge and high genius; even Stirling gives it as his opinion that Newton has not chosen the best manner of treating the problem. Nevertheless I am inclined not only to think that Newton's way indeed is the easiest and best, but even to suppose that he regarded his solution as all but self-evident; at least the proof, when undertaken in the right way, is very easy, as we shall see.

Instead of giving a copy of the latin text or a verbatim translation of the said lemma, I have preferred to give a sort of paraphrase. The only alterations made are: 1° that I have omitted the first case (equidistant arguments) as only special, 2° that instead of Newton's *geometrical* language and notation I have used the now commonly used analytical expressions, and a notation in which the arguments are denoted by small letters, the corresponding values of the functions by capitals, and the *divided differences* by a δ with dashes indicating their order, and followed by the letters (in brackets) indicating the arguments on which the difference depends. I hope that these alterations may be found inmaterial as to the meaning, and of some assistance to the reader. The paraphrase runs thus:

LEMMA V.

" Of an unknown function we know $(n+1)$ values, A, B, C, D, E . . . ., " corresponding to the arguments a, b, c, d, e . . . .; let it be required " to represent these values by an integral and rational algebraical function, " so that we shall be able for any argument x to calculate directly the " corresponding value X of the said algebraical function."

" *Solution.* Form the complete system of divided differences in the following manner:

$$
\begin{array}{llllll}
a & A & & & & \\
 & & \delta'(a,b) & & & \\
b & B & & \delta''(a,b,c) & & \\
 & & \delta'(b,c) & & \delta'''(a,b,c,d) & \\
c & C & & \delta''(b,c,d) & & \delta^{iv}(a,b,c,d,e) \\
 & & \delta'(c,d) & & \delta'''(b,c,d,e) & \\
d & D & & \delta''(c,d,e) & & \cdots\cdots \\
 & & \delta'(d,e) & & \cdots\cdots & \\
e & E & & \cdots\cdots & & \\
 & & \cdots\cdots & & & \\
\vdots & \vdots & & & &
\end{array}
$$

" The divided first differences are found by taking the difference of any " two adjacent values of the function and dividing it by the difference of

* This is the *epitheton ornans*, by which Gauss distinguishes Newton, and, as far as I know, nobody else.

"the corresponding arguments; $\delta'(a,b)=\dfrac{\mathrm{A}-\mathrm{B}}{a-b}$, $\delta'(b,c)=\dfrac{\mathrm{B}-\mathrm{C}}{b-c}$, and "so on.

"The divided differences of any *higher* order are found by taking the "difference of any two adjacent differences of the preceding order, and "dividing it by the difference of the two arguments which are not common "to the said two differences: thus

$$\delta''(a,b,c)=\frac{\delta'(a,b)-\delta'(b,c)}{a-c},\ \delta''(b,c,d)=\frac{\delta'(b,c)-\delta'(c,d)}{b-d},\ \text{\&c.}$$

$$\delta'''(a,b,c,d)=\frac{\delta''(a,b,c)-\delta''(b,c,d)}{a-d},\ \delta'''(b,c,d,e)=\frac{\delta''(b,c,d)-\delta''(c,d,e)}{b-e},\text{\&c.}$$

"This done, we shall have

$$\mathrm{X}=\mathrm{A}+(x-a)\times$$
$$\{\delta'(a,b)+(x-b)\big[\delta''(a,b,c)+(x-c)[\delta'''(a,b,c,d)+(x-d)[\delta^{\mathrm{iv}}(a,b,c,d,e)+\,..]]\big]\}"$$

Before I give the proof, I think it well (though it may be unnecessary) to draw the attention of the younger readers to the following points:—

1°. It matters nothing in what way we make the subtraction, provided only that this way is the same in the dividend and in the divisor: (of course, $\dfrac{\mathrm{A}-\mathrm{B}}{a-b}=\dfrac{\mathrm{B}-\mathrm{A}}{b-a}$).

2°. If F denotes an integral and finite algebraical function of one argument or of more arguments, then not only is $\mathrm{F}(p)-\mathrm{F}(q)$ divisible by $(p-q)$ and the quotient of a lower degree than that of F, but $\mathrm{F}(p,r,s,t\ldots.)-\mathrm{F}(q,r,s,t\ldots.)$ is also divisible by $(p-q)$, and the quotient of a lower degree than that of F; this is easily shown by putting $\mathrm{F}(p,r,s,t\ldots.)=k_0+k_1p+k_2p^2+k_3p^3+\ldots..$ and $\mathrm{F}(q,r,s,t\ldots.)=k_0+k_1q+k_2q^2+k_3q^3+\ldots$, which is always permitted, ($k_0, k_1, k_2, k_3,\ldots$ being functions of $r,s,t,\ldots.$)

3°. Hence it follows, since A, B, C, ... are functions of the nth degree, that the divided first differences are of the $(n-1)$th degree, the divided second differences are of the $(n-2)$th degree, and so on, until the divided nth difference, which being of the $(n-n)$th degree, must be a constant, independent of the arguments. Of course all differences of a degree higher than the nth, must vanish.

To prove that the value above assigned to X is correct, we have only to write x, X, and the several divided differences, at the top of the scheme given above; when we have

$$\delta'(x,a)=\frac{\mathrm{X}-\mathrm{A}}{x-a},\ \delta''(x,a,b)=\frac{\delta'(x,a)-\delta'(a,b)}{x-b},$$
$$\delta'''(x,a,b,c)=\frac{\delta''(x,a,b)-\delta''(a,b,c)}{x-c},\ \text{\&c.}$$

Hence

$$\mathrm{X}=\mathrm{A}+(x-a)\delta'(x,a),$$
$$\delta'(x,a)=\delta'(a,b)+(x-b)\delta''(x,a,b),$$
$$\delta''(x,a,b)=\delta''(a,b,c)+(x-c)\delta'''(x,a,b,c),$$

and so on until $\delta^{(n)}(x,a,b,c\ldots\ldots)=\delta^{(n)}(a,b,c,d\ldots\ldots)$

$\therefore$ (by substitution) $X=A+(x-a)\times$

$$\{\delta'(a,b)+(x-b)[\delta''(a,b,c)+(x-c)[\delta'''a,b,c,d)+(x-d)[\delta^{\text{iv}}(a,b,c,d,e)+\ldots]]]\}$$

4th April, 1869. LUDV. OPPERMANN.

*** We have reason to expect that this is only the first of a series of contributions from the distinguished Corresponding Member of the Institute at Copenhagen.—ED. *J. I. A.*

CORRESPONDENCE.

ON A TABLE FOR FACILITATING THE VALUATION OF ABSOLUTE REVERSIONS.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—The accompanying Table, suggested by Mr. Sprague's Table of the Value of Life Interests (contained in the 8th volume of your *Journal*) will, I think, be found useful. Its title is sufficient explanation of the purpose for which it is intended.

The values indicated are based upon the well known formula $v-(1-v)a$, in which v is the present value of £1 due a year hence, and a the price of a whole-life annuity of £1; and it is evident that, whilst dealing with the same rate of interest, the difference between the results for any two given annuity prices consists of the difference between such prices multiplied by $(1-v)$. If a series of annuity prices be taken in arithmetical progression, the second term of the formula will form a series in like progression, causing the results of the whole expression to diminish by a constant quantity. When, therefore, the values for two prices are known, the value for any intermediate price can be found by the most simple method of interpolation.

If a be successively increased by one shilling, the series of values, starting from that corresponding to an annuity costing x pounds, will be—

	£	s.	d.	
for	x	0	0	$v-(1-v)x$
,,	x	1	0	$v-(1-v)x-(1-v)\frac{1}{20}$
,,	x	2	0	$v-(1-v)x-(1-v)\frac{2}{20}$
,,	x	3	0	$v-(1-v)x-(1-v)\frac{3}{20}$
	....			....

And similarly, if the successive increase be one penny, we shall have